AIAA 81-4139

Numerical Solution of Simplified Oswatitsch Equation in Transonic Flow

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I. Introduction

THE simplified Oswatitsch equation is a one-dimensional nonlinear singular integral equation, occurring in transonic aerodynamics.[1] It is an approximate version of Oswatitsch's integral equation, whose kernel has a dipole singularity. According to transonic small perturbation theory, the Oswatitsch equation is the basic equation governing the steady inviscid irrotational flow of a perfect gas past a thin symmetric profile at zero incidence, with subsonic freestream Mach number $M_\infty < 1$ and a local supersonic region near the maximum thickness of the body. The Oswatitsch integral equation and the simplified version are valid for shock-free transonic flow as well as for flows with a shock discontinuity. The study of the simplified Oswatitsch equation is of much interest, because converged solution of it can serve as a good starting solution for finite-difference relaxation procedures for solving problems of transonic aerodynamics.

The simplified Oswatitsch equation was first solved by Oswatitsch[1] using rough collocation procedure. The solutions predicted all the typical transonic features, although quantitatively they were of moderate accuracy. Subsequently, a number of authors attempted to obtain improved solutions of the Oswatitsch equation[2,3] by introducing various approximations leading to nonlinear integral equations similar to the simplified Oswatitsch equation. Among these, the works of Nørstrud[4] and Nixon[5] deserve special mention.

In the present work, the simplified Oswatitsch equation has been solved by two different numerical procedures, viz., the direct iteration scheme (DIS) proposed by Niyogi and Chakraborty[6] and by the recent perturbed iterative scheme (PIS) put forward by Dey[7] for solving a system of nonlinear algebraic equations. Further, this nonlinear integral equation was used as a test case for studying the global convergence behavior of PIS. From computational results, it has been found that for a parabolic arc profile there exists a range of values for the reduced thickness ratio τ (which is a transonic similarity parameter), where both the procedures lead to the same shock-free supercritical solution, and that in this range PIS converges much faster than DIS. However, for higher values of τ beyond this range, there exists another range where, contrary to expectations, DIS converges but PIS fails to converge, indicating that DIS has a wider range of convergence.

II. Formulation of the Problem

The simplified Oswatitsch equation[2,3]

$$u(x) = u_L(x) + \tfrac{1}{2}u^2(x) - \int_0^1 \frac{u^2(\xi)}{2b} E\left(\frac{\xi - x}{b}\right) \mathrm{d}\xi \qquad (1)$$

which appears in transonic aerodynamics is a nonlinear singular integral equation. Here, $u(x)$ is the unknown function and $u_L(x)$ is a known function. The kernel E is given by

$$E(z) = \frac{4}{\pi(1+z^2)^5}\left[\frac{\pi}{2}(5 - 10z^2 + z^4)\,|z| - (1 - 10z^2 + 5z^4)\ln|z| - \frac{1}{12}(25 - 71z^2 - z^4 - z^6)(1+z^2)\right] \qquad (2)$$

which possesses a logarithmic singularity at $\xi = x$. In terms of transonic reduced variables, u represents the reduced velocity component parallel to the freestream direction x, the reduced velocity components (u, v) being defined in terms of their true values, denoted by capital letters by[2]

$$M_\infty < 1, \quad x = X, \quad u = (U - U_\infty)/(c^* - U_\infty)$$
$$y = Y(1 - M_\infty^2)^{1/2}, \quad v = V/[(c^* - U_\infty)(1 - M_\infty^2)^{1/2}] \qquad (3)$$

Here X and Y denote the body-fixed rectangular Cartesian coordinate system, with origin situated at the tip of the profile, M_∞ is the freestream Mach number (defined by U_∞ / c_∞, where U_∞ and c_∞ denote undisturbed flow speed and sound speed, respectively) and c^* the critical speed of sound.

Equation (1) is an approximate version of Oswatitsch equation, which is a two-dimensional nonlinear singular integral equation governing the steady plane inviscid irrotational transonic flow past a thin symmetric profile at zero incidence with freestream Mach number $M_\infty < 1$. The solution of the corresponding linearized problem is denoted by $u_L(x)$. For an airfoil shape in the form of a parabolic arc, the linearized solution may be expressed in reduced coordinates as[2]

$$u_L(x) = \frac{4\tau}{\pi}\left[1 + \left(x - \frac{1}{2}\right)\ln\left|\frac{1-x}{x}\right|\right] \qquad (4)$$

where τ is a transonic similarity parameter representing the reduced thickness ratio of the profile. The quantity b appearing in Eq. (1) is a parameter, whose value for a parabolic arc profile may be taken to be $2/\pi$, as suggested by Oswatitsch.[1] It should be noted that according to the reduction Eq. (3), the flow at a point is subsonic for $u < 1$ and supersonic for $u > 1$. Further, the reduced velocity component u is not a small quantity, and is of the order of unity in the transonic range. Our object is to solve Eq. (1) numerically, in the typical transonic range.

III. Method of Solution

Equation (1) has been solved here numerically by two different procedures, viz., the direct iteration scheme[6] and by the perturbed iterative scheme.[7] The DIS is given by

$$u_{n+1}(x) = u_L(x) + \tfrac{1}{2}u_n^2(x) - \int_0^1 \frac{u_n^2(\xi)}{2b} E\left(\frac{\xi - x}{b}\right)\mathrm{d}\xi, \qquad n = 0,1,2,3,\ldots \qquad (5a)$$

with the starting solution

$$u_0(x) = (\sqrt{3}+1)\,[1 - \{1 - (\sqrt{3}-1)\,u_L(x)\}^{1/2}] \qquad (5b)$$

which is known[8] to be a good approximate solution of Eq. (1).

To evaluate the singular integral in Eq. (5a), it appears convenient to break-up the interval $[0,1]$ into k number of subintervals. The unknown u is taken to be a step function in

Received Nov. 18, 1980.

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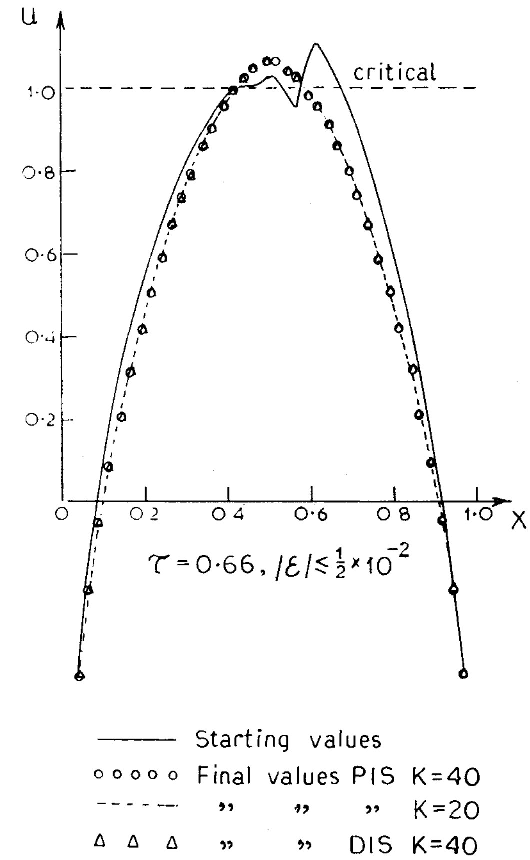

Fig. 1 Parabolic arc profile in supercritical transonic flow.

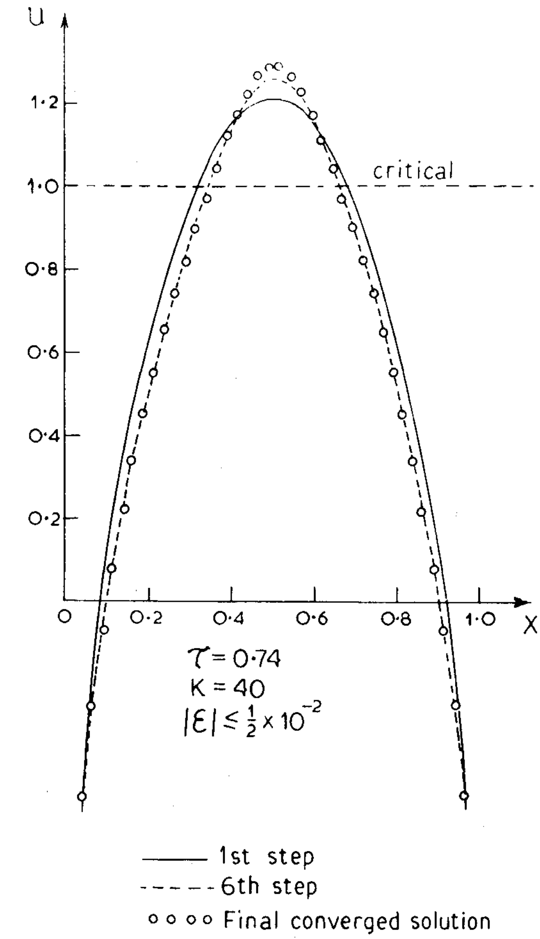

Fig. 2 Solution by DIS converged in 12 steps.

Table 1 Number of iteration steps for convergence of PIS and DIS

Reduced thickness ratio	PIS	DIS
0.60	3	8
0.62	4	9
0.64	5	9
0.66	6	10
0.70	Does not converge	11
0.74	Does not converge	12

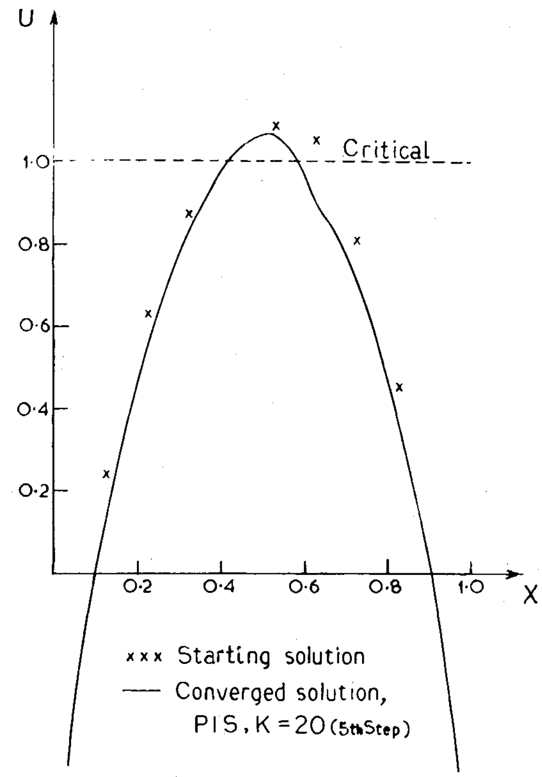

Fig. 3 Parabolic arc profile, $\tau = 0.66$.

these subintervals, being equal to the value at the middle of the respective subintervals. Noting that the integration of the function $E(z)$ may be carried out in closed form, the integral in Eq. (5a) is approximated by a finite sum over the pivotal points in the form

$$\int_0^1 \frac{u_n^2(\xi)}{2b} E\left(\frac{\xi - x}{b}\right) \mathrm{d}\xi = \frac{1}{2b} \sum_{i=1}^{k} (u_{ni})^2 g_i(x) \tag{6}$$

where u_{ni} denotes the value of u_n at the center of the ith subinterval. The explicit form of the function $g_i(x)$ may be found in Refs. 2 and 3, so that it need not be reproduced here. It is to be noted that through integration, the logarithmic singularity at $\xi = x$ has been removed and $g_i(x)$ is free from singularity.

For applying the PIS, we approximate the integral in Eq. (1) by a finite sum corresponding to Eq. (6). Then denoting by the suffix j, the value of a function at a pivotal point $x = x_0 + jh$, $j = 1,2,\ldots,k$, we obtain the following approximate system, in place of the integral Eq. (1):

$$U_j = (u_L)_j + \frac{1}{2} U_j^2 - \frac{1}{2b} \sum_{i=1}^{k} U_i^2 g_{ij}, \qquad j = 1,2,\ldots,k \tag{7}$$

where U_j denotes the value of u at the pivotal point j. Thus we have a system of k-nonlinear algebraic equations for the k-

unknowns U_j, $j=1,2,\ldots,k$. The resulting system is then solved by PIS, which is believed to have quadratic rate of convergence.

IV. Numerical Results and Discussion

Using DIS and PIS numerical solutions have been computed for a thin symmetric parabolic arc profile and an NACA 0012 profile. The approximate solution Eq. (5b) may also be taken as an initial guess at the solution for the perturbed iterative scheme. Moreover, solutions were also computed with starting solutions other than Eq. (5b), such as

$$u_0(x)=2[1-\{1-u_L(x)\}^{1/2}] \tag{8a}$$

$$u_0(x)=u_L(x) \tag{8b}$$

and

$$u_0(x)=1 \tag{8c}$$

It is found from computations for the case of a parabolic arc profile, that the flow is entirely subsonic for values of the reduced thickness ratio $\tau<0.64$, while it is mixed subsonic supersonic for $\tau>0.64$. In the subsonic range, both PIS and DIS converge very rapidly, requiring less than 10 iteration steps, for results correct to two decimal places. Of these two methods, PIS converges in almost one-half or one-third the number of steps compared to DIS, to the same solution irrespective of the starting solutions Eq. (5b) or Eqs. (8a-c). Further, the super-critical continuous (i.e., shock-free) solution is found in the range

$$0.64\leq\tau<0.68 \tag{9}$$

by both methods and they converge to the same super-critical solution, irrespective of the starting solution. Further, computations carried out with 40 and 80 pivotal points for the same profile shape with the same reduced thickness ratio show that there is not mentionable difference in the converged solution for the two choices of the number of pivotal points. However, beyond the range of Eq. (9) there exists another range

$$0.68<\tau<0.74 \tag{10}$$

where DIS converges to a continuous (i.e., shock-free) supercritical solution, with any of the starting solutions Eq. (5b) or Eqs. (8a-c). Contrary to expectations, PIS fails to converge in this range. For values of τ greater than 0.74, an expansion shock seems to be formed at the accelerating sonic point and, ultimately, the whole computation diverges.

To test if any solution with shock discontinuity exists, we take a discontinuous solution as the starting solution for values of τ in the range of Eq. (9). However, PIS does not converge with such starting solutions, whereas DIS converges to a smooth solution shown in Fig. 1. The converged solution, correct to two decimal places obtained by DIS, for a parabolic arc profile with reduced thickness ratio $\tau=0.74$, is shown in Fig. 2. Only 12 iteration steps are needed for convergence. For $\tau=0.66$, with 20 pivotal points, a peculiar feature is to be seen when PIS converges in 5 steps to a slightly asymmetric flow, shown in Fig. 3. This asymmetry is ascribed to the larger truncation error in the case of a smaller number of pivotal points.

The number of iteration steps needed for convergence, correct to two decimal places, for a parabolic arc profile with 40 pivotal points is shown in Table 1, which clearly shows the faster rate of convergence of PIS.

In the case of an NACA 0012 profile also, a range of freestream Mach numbers have been found where both the schemes converge to a shock-free supercritical solution.

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AIAA 81-4140

H-R_x Method for Predicting Transition

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Introduction

THIS Note gives a brief description of a shortcut method, the H-R_x method, for predicting transition in a wide class of boundary-layer flows, including the effects of pressure gradient, surface heat transfer, and suction. Here H and R_x are the body shape factor and the Reynolds number, respectively, based on distance x measured in the direction of the flow. The method is extremely simple to use and a good substitute to the well known but rather complicated e^9 method.

The e^n Method of Forecasting Transition

The H-R_x method has not been correlated with test data but rather has been justified in terms of Tollmien-Schlichting waves and e^9 type calculations. Data are gradually being accumulated that show that the e^9 method is the best all-around method that now exists for predicting boundary-layer transition. Since the H-R_x method is rooted in the e^9 method, we proceed to make some comments about the latter.

Transition, although it may commence with the amplification of Tollmien-Schlichting waves as described by linear stability theory, is dominated in its late stages by three dimensional and nonlinear effects. Why then does transition

Received Oct. 6, 1980; revision received Feb. 3, 1981.

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